CHAPTER XIII

PROPELLERS

13·0. In this chapter we consider the elementary theory of the screw propeller, and endeavour to show what assumptions are usually made in arriving at methods for numerical calculation.

13·1. Propellers. A propeller consists of a certain number of *blades* rotating about an axis.

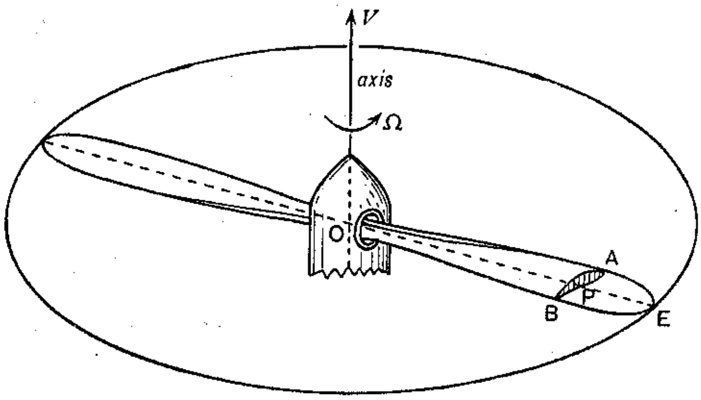

FIG. 13·1.

Propellers are designed to exert thrust to drive the aircraft forward. If T is the thrust in the direction of the axis of rotation, Ω the angular speed of the propeller shaft, Q the torque exerted by the engine and V the forward speed in the direction of the axis of rotation, the work done per unit time by the engine is $Q\Omega$ and by the propeller is TV. Thus a propeller converts torque power into thrust power and the *efficiency* is

$$\eta = \frac{TV}{Q\Omega}.$$

Thrust is obtained by proper shaping of the blades, which are in fact twisted aerofoils.

Every point of a blade lies on a circular cylinder whose axis is the axis of rotation, and therefore as the aircraft advances each point describes a helix or spiral curve on the cylinder on which that point always lies. Of these cylinders there is one of maximum radius. The point of the propeller blade which lies on this cylinder is the *tip*, E in fig. 13·1, of the blade. From the tip E we can draw a perpendicular OE to the axis of rotation. This line may be called the *axis of the blade*. The section of the cylinder by the plane through OE perpendicular to the axis of rotation is called the *propeller disc*, or simply the disc.

If we take a point P on OE such that $OP = r$ and describe a cylinder whose axis is the axis of rotation and whose radius is OP, the points of the surface of the blade which lie on this cylinder will constitute a curve resembling an aerofoil profile; the totality of such curves defines the shape of the blade. It is, however, customary to define the shape of the blade by plane sections. Thus at P the section of the blade will be that made by the plane through P perpendicular to the blade axis, giving, for example, the profile marked AB in fig. 13·1. Such a section is called a *blade profile*.

The portion of a blade between the blade profiles at distances r and $r + dr$ from the axis of rotation is called a *blade element*.

13·2. How thrust is developed. Each blade element behaves like an aerofoil and undergoes lift and drag, and of course leaves a wake behind as it moves. In the present section we shall omit all consideration of the velocity induced by the wake as we are here concerned only with a main principle.

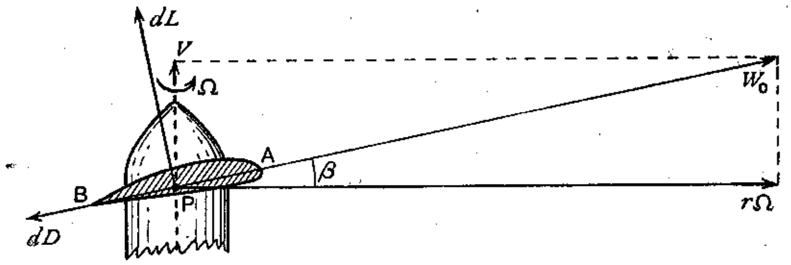

FIG. 13·2.

Fig. 13·2 shows the blade element one of whose bounding profiles is the blade profile of fig. 13·1.

We now introduce two assumptions which will be retained throughout this chapter.

Assumption I. The aircraft is moving in the direction of the axis of rotation of the propeller.

This is only rigorously true for a certain particular incidence of the main lifting system.

Assumption II. Every point of the blade element between the planes r and $r + dr$ has, due to the rotation, a velocity $r\Omega$.

This is clearly the more nearly true, the greater the value of r.

The resultant velocity is then $\mathbf{W}_0$ where $W_0^2 = V^2 + r^2\Omega^2$.

If then the blade profile is disposed as in the figure it will undergo a lift dL perpendicular to $\mathbf{W}_0$ and a drag dD opposed to $\mathbf{W}_0$.

If β is the angle which $\mathbf{W}_0$ makes with the direction of $r\Omega$ the blade element has a thrust in the direction of $\mathbf{V}$ of amount

(1) $$dT = dL \cos \beta - dD \sin \beta,$$

and the whole blade undergoes a forward thrust equal to the sum of the dT arising from its various elements.

At the same time there is a torque

(2) $$dQ = r(dL \sin \beta + dD \cos \beta)$$

opposing the rotation of the blade, and the sum of the dQ is the total torque which the engine must exert to turn the blade.

It appears from this elementary exposition that, if R is the radius of the propeller disc, the maximum speed of a blade profile is $\sqrt{(V^2 + R^2\Omega^2)}$ and that, if compression waves are not to develop, this maximum must be kept below the speed of sound (see 15·5). This places a limitation on the radius of the propeller disc when maximum values of V and Ω are assigned. Again, in order that the speed of sound may be more nearly approached without adverse effects the tip profiles must be made thin (see 15·7). This is also dynamically desirable to avoid too great a thickness at the root of the blade which, for reasons of strength, would be necessary if the blade were unduly massive towards the tip.

We also note that if $\Omega = 0$, then $\mathbf{W}_0 = \mathbf{V}$, $\beta = 90°$ and $dQ = r\,dL$. Thus, if the blade profile is set so that its axis of zero lift is in the direction of $\mathbf{V}$, we shall have $dQ = 0$ and there will be no torque on the propeller shaft from this element. If all the blade profiles are set in this way the propeller is said to be *feathered*. The feathered attitude is usually a possible setting with propellers of variable pitch (see 13·42).

In postulating the existence of lift and drag on the blade elements we are tacitly assuming that there is circulation round these elements, and therefore that the surface of the blade is equivalent to a sheet of bound vortices. These will give rise to a wake and therefore to induced velocity additional to $\mathbf{W}_0$. We proceed to discuss the nature of the wake.

13·3. The slipstream. This is constituted by the air which has passed through the propeller disc.

Consider the portion of the blade between P on the blade axis and the tip E. This portion behaves as an aerofoil, and, if we adopt the lifting line theory, from the trailing edge PE there escapes a vortex sheet. As the blade rotates this sheet assumes a helicoidal or spiral form, as indicated in the figure, and the slipstream consists of an assemblage of such surfaces, one for each blade. In the wake of the trailing edge there is therefore a downwash, and just ahead there is an upwash from the bound vortices. As the air passes through the propeller disc its axial velocity must be continuous. The downwash velocity at a point of the wake has also a component tangential to the cylinder on which that point lies and the air in the slipstream is therefore rotating about the line of advance of the aircraft in the same sense as the angular velocity of the propeller.

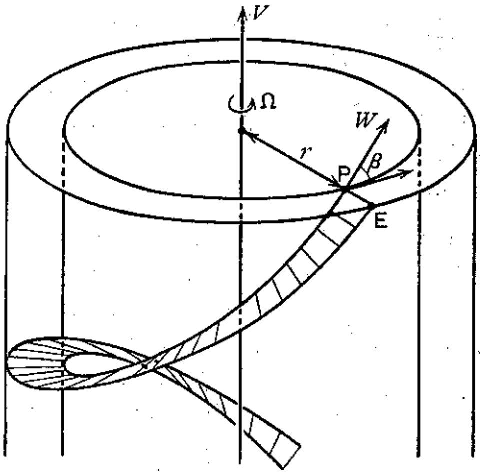

FIG. 13·3 (*a*).

Ahead of the propeller disc the bound vortices induce a tangential component in the opposite sense, but the undisturbed air into which the propeller disc is advancing can have no axial rotation, so this must be cancelled by the velocity induced by the wake.

The vortex sheet which springs from the trailing edge of a blade is unstable and rolls up into a spiral of concentrated vorticity. In the particular case in which the lift is uniformly distributed along the blade axis, the vortex system due to the blade will consist of (i) a bound vortex along the blade axis, (ii) a spiral vortex springing from the tip of the blade, and (iii) a rectilinear vortex along the part of the axis of rotation which is on the downstream side of the disc. (See the frontispiece.)

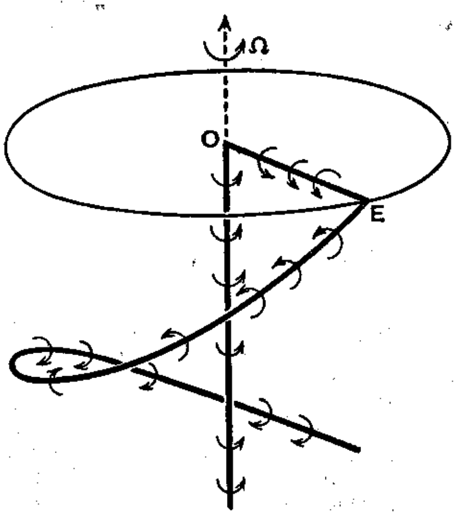

FIG. 13·3 (*b*).

Fig. 13·3 (*b*) illustrates the scheme, and in the case mentioned the circulation round each part has the same value Γ, say. When the lift is not uniformly distributed along the blade axis, the sheet will still roll up almost immediately into concentrated vortices * whose cores will be represented by lines of the spiral type (ii) and the axial type (iii). It is this arrangement which replaces the horseshoe vortices of the lifting line theory of aerofoils. The vortex (ii) has been described as spiral in form. This must not

* J. Valensi, Étude de l'air autour d'une hélice, *Thèse*, Paris (1935), has obtained some beautiful photographs of the vortices and developed a method of obtaining quantitative results from their measurement.

be taken to mean that it is a regular helix drawn on a circular cylinder. For some calculations it is convenient to make that an assumption, but in general the diameter of the slipstream contracts (see fig. 13·31) as we proceed downstream and the slipstream only ultimately assumes the cylindrical form in an ideal incompressible inviscid fluid.

Observe that if there are several blades,* each will contribute a vortex of the type (iii) so that in the case of B blades, each with circulation Γ, the axial vortex will be of circulation $B\Gamma$.

13·31. Velocity and pressure in the slipstream. When the propeller advances with constant velocity $\mathbf{V}$ and rotates with constant angular velocity $\mathbf{\Omega}$, the motion of the air at a point fixed in space is not steady, the pressure p and the velocity $\mathbf{q}$ depend on the time. If, however, we take a system of axes of reference fixed in the propeller, and therefore rotating and advancing with it, the motion is steady with regard to these axes. If $\mathbf{q}'$ is the air velocity measured with respect to these moving axes, Bernoulli's equation (see 2·11) becomes

$$(1) \qquad \frac{p}{\rho} + \tfrac{1}{2}q'^2 - \tfrac{1}{2}\Omega^2 r^2 = \text{constant},$$

the last term on the left representing the potential energy of the fictitious field of force introduced by the rotation. If we denote by (q_a, q_r, q_t) the axial, radial, and tangential components of the absolute air velocity $\mathbf{q}$, the components of the relative velocity will be $(q_a, q_r, q_t - r\Omega)$ and therefore

$$q'^2 = q_a^{\,2} + q_r^{\,2} + (q_t - r\Omega)^2 = q^2 + r^2\Omega^2 - 2q_t r\Omega$$

and so (1) becomes

$$(2) \qquad \frac{p}{\rho} + \tfrac{1}{2}q^2 - q_t r\Omega = \text{constant},$$

where q is now the absolute air speed.

Fig. 13·31 shows schematically a section through the centre O of the propeller disc DD', the hatched part representing the slipstream, the point of view being that of an observer moving with the axes of reference. Outside the slipstream the motion is everywhere irrotational. Far ahead of the propeller the air appears to form a uniform stream $-\mathbf{V}$, as it also does far astern *except in the slipstream.*

The slipstream itself contracts from its greatest diameter DD' at the disc, and asymptotically approaches the form of a circular cylinder typified by the diameter GG' in fig. 13·31.

It is useful to regard the air ahead of the disc and bounded by the surface whose sections are indicated by ADG, $A'D'G'$ in fig. 13·31 as an "extension"

* See the frontispiece. The photograph was taken in water in a cavitation tank with the propeller axis horizontal. The vortices are made visible by the cavitation bubbles (which contain water vapour) escaping from the blades.

of the slipstream, but it should be carefully noted that in crossing say DG from inside to outside the slipstream there is an abrupt change of velocity so that the boundary of the actual slipstream is a vortex sheet, whereas in crossing AD

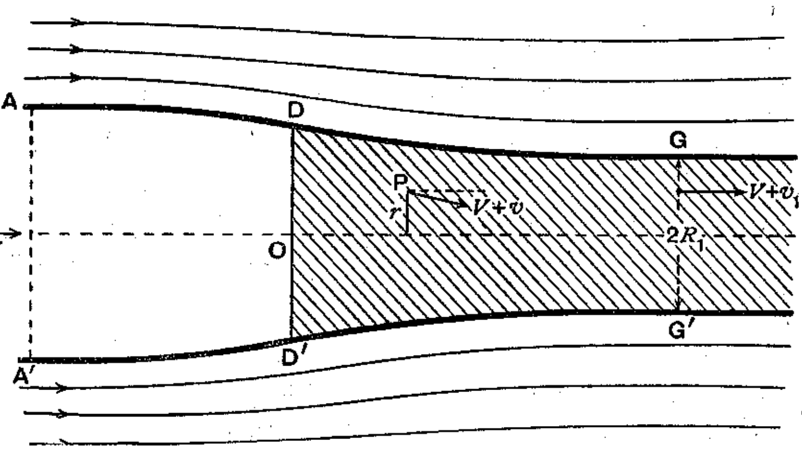

FIG. 13·31.

there is no abrupt change and therefore the boundary of the above "extension" is not a vortex sheet.

Now consider a point P of the slipstream. Let us put $q_t = r\omega$, so that ω is the angular speed of a plane containing the air particle which is at P, and the axis of propeller rotation. Then outside the slipstream $\omega = 0$, for the motion is irrotational and there can therefore be no circulation. Thus (2) can be written

$$\frac{p}{\rho} + \tfrac{1}{2}q^2 - r^2\omega\Omega = \frac{\Pi}{\rho} + \tfrac{1}{2}V^2, \tag{3}$$

where Π is the pressure at infinity ahead of the disc; and so (3) is valid throughout the field of flow.

Finally, if the projection of $\mathbf{q}$ on the plane of the section of fig. 13·31 is denoted by $(V + v)$ we have $q^2 = (V + v)^2 + r^2\omega^2$ and therefore

$$\frac{p - \Pi}{\rho} = r^2\omega\left(\Omega - \frac{\omega}{2}\right) - v\left(V + \frac{v}{2}\right). \tag{4}$$

This is an exact equation which applies throughout the fluid.

If we describe the position of P by cylindrical coordinates (r, θ, z) where z is the distance downstream from O, the pressure p and the velocities v, ω are functions of all three coordinates.

13·4. Interference velocity. The trailing vortex system described in 13·3 gives rise to induced velocity, known as *interference velocity*.

This velocity will have three components, axial, radial, and tangential, the latter term referring to the tangent to that circular section of the slipstream which passes through the point which we are considering.

Assumption III. The radial component may be neglected.

In the plane of the disc, on the downstream face, the interference velocity will, see fig. 13·3 (*a*), have its axial component opposite to the direction of advance of the propeller and its tangential component in the same sense as the rotation.* Thus relatively to the propeller the total axial component is increased to, say, $V(1 + a)$, and the total tangential component is decreased to, say, $r\Omega(1 - a')$. The numbers a and a' are called *interference factors*. For a propeller they are both positive.

Now consider points P_2, P_1 at radial distance r just ahead and just astern of the disc. By symmetry the bound vortices induce no axial velocity at these points, so the total induced axial velocity is the same at P_1 and P_2 and may be written

$$v = aV = \tfrac{1}{2}v_1.$$

If the bound vortices induce angular velocity ω' at P_1, by symmetry, they must induce angular velocity $-\omega'$ at P_2. Thus the total induced angular velocity can be written as $\frac{1}{2}\omega_1 + \omega'$ at P_1 and $\frac{1}{2}\omega_1 - \omega'$ at P_2. But P_2 is in the irrotational flow and therefore $\frac{1}{2}\omega_1 - \omega' = 0$. Therefore the total angular velocity induced at P_1 by both bound and free vortices is $\omega_1 = 2\omega' = 2a'\Omega$.

13·41. The force on a blade element. To calculate the force we introduce

Assumption IV. Each blade element may be treated as a two-dimensional aerofoil moving with the relative velocity calculated at the downstream face of the disc.

The relative velocity here referred to is the velocity whose axial and radial components at distance r are $V(1 + a)$, $r\Omega(1 - a')$.

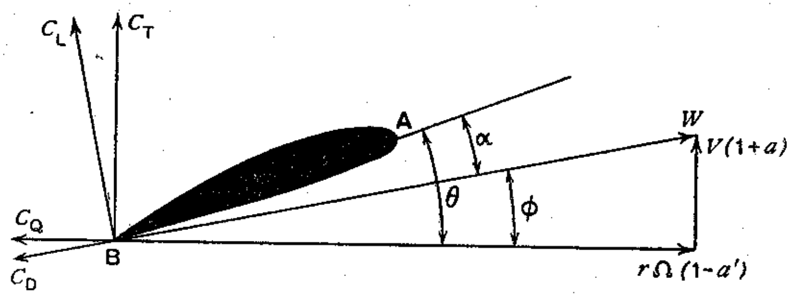

FIG. 13·41.

Let AB be the chord of the blade element.

The angle θ which AB makes with the direction of the tangential velocity is called the *blade angle*. If ϕ is the angle which the relative air velocity W makes with the tangential velocity the incidence is $\alpha = \theta - \phi$, and the cor-

* On the upstream face the tangential component vanishes, see 13·3. The axial component is the same on both faces.

responding lift and drag coefficients may be found from the graphs appropriate to the blade profile. If c is the chord, the lift and drag in the blade element are

$$dL = C_L \cdot \tfrac{1}{2}\rho W^2 c\, dr, \qquad dD = C_D \cdot \tfrac{1}{2}\rho W^2 c\, dr,$$

where C_L and C_D are the lift and drag coefficients of the blade element.

If we now write

$$(1) \qquad C_T = C_L \cos\phi - C_D \sin\phi, \qquad C_Q = C_L \sin\phi + C_D \cos\phi,$$

we get for the thrust and torque due to the blade element

$$dT^{(1)} = C_T \cdot \tfrac{1}{2}\rho W^2 c\, dr, \qquad dQ^{(1)} = C_Q \cdot \tfrac{1}{2}\rho W^2 cr\, dr.$$

If there are B blades the contributions of all the blade elements at distance r will be

$$dT_r = BdT^{(1)}, \quad dQ_r = BdQ^{(1)}.$$

The projected area of all the blade elements on their chords is $Bc\,dr$ and the area of the annulus between radii r and $r + dr$ is $2\pi r\, dr$. The ratio of these areas is termed the *solidity* of the blade element and is denoted by $\sigma = Bc/(2\pi r)$. Also, from fig. 13·41,

$$(2) \qquad \tan\phi = \frac{V}{r\Omega} \cdot \frac{1 + a}{1 - a'},$$

and therefore

$$(3) \quad \frac{dT_r}{dr} = C_T \cdot \pi\rho\sigma r V^2 (1 + a)^2 \operatorname{cosec}^2\phi = C_T \cdot \pi\rho\sigma r^3 \Omega^2 (1 - a')^2 \sec^2\phi,$$

$$(4) \quad \frac{dQ_r}{dr} = C_Q \cdot \pi\rho\sigma r^2 V^2 (1 + a)^2 \operatorname{cosec}^2\phi = C_Q \cdot \pi\rho\sigma r^4 \Omega^2 (1 - a')^2 \sec^2\phi.$$

13·42. Characteristic coefficients. If we write

$$(1) \qquad T_r = \tau_\xi \rho R^4 \Omega^2, \qquad Q_r = \kappa_\xi \rho R^5 \Omega^2, \qquad \xi = \frac{r}{R},$$

equations (3) and (4) of 13·41 become

$$(2) \quad \frac{d\tau_\xi}{d\xi} = C_T \pi\sigma\xi^3 (1 - a')^2 \sec^2\phi, \qquad \frac{d\kappa_\xi}{d\xi} = C_Q \pi\sigma\xi^4 (1 - a')^2 \sec^2\phi.$$

The thrust and torque on the whole propeller are then

$$(3) \qquad T = \tau\rho R^4 \Omega^2, \qquad Q = \kappa\rho R^5 \Omega^2, \quad \text{where}$$

$$(4) \qquad \tau = \int_0^1 \frac{d\tau_\xi}{d\xi}\, d\xi, \qquad \kappa = \int_0^1 \frac{d\kappa_\xi}{d\xi}\, d\xi.$$

In terms of n, the number of revolutions per unit time, and D, the diameter of the disc, we define the *rate of advance coefficient*

$$(5) \qquad J = \frac{V}{nD} = \frac{\pi V}{R\Omega} = \pi\xi \frac{1 - a'}{1 + a} \tan\phi,$$

the last term being obvious from an inspection of fig. 13·41.

The efficiency of the propeller is then

$$(6) \qquad \eta = \frac{TV}{Q\Omega} = \frac{J\tau}{\pi\kappa}.$$

For a given propeller the geometrical quantities σ, θ are known for each value of ξ, and also the aerodynamic quantities C_L, C_D for each value of α. Thus, if we know the interference factors a, a' (see 13·7), we can calculate ϕ, J, C_T, C_Q and so obtain the differential coefficients $d\tau_\xi/d\xi$, $d\kappa_\xi/d\xi$ from (2).

Graphical integration will now give *the characteristic coefficients* τ, κ and therefore η for different values of J. Typical graphs are shown in fig. 13·42.

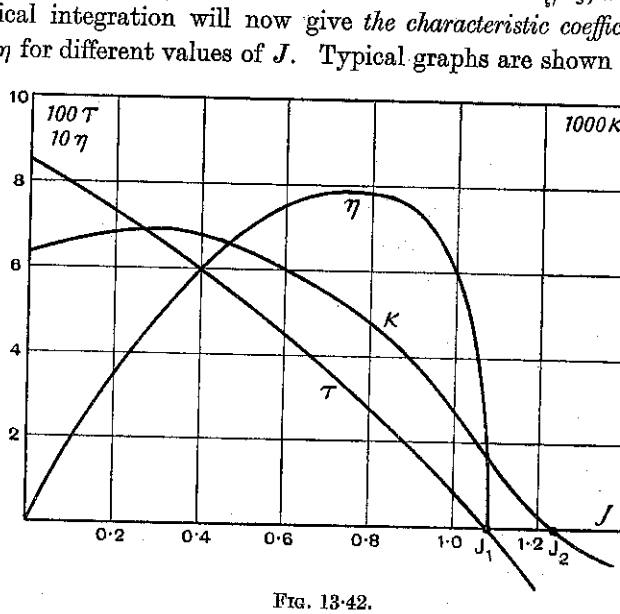

FIG. 13·42.

We observe from (6) that if τ vanishes for a value J_1 of J, η will also vanish for this value, and the graph shows that κ vanishes for a value J_2 of J.

If $J = J_1$, there is no thrust and the propeller is feathered.

If $J_1 < J < J_2$, τ and therefore the thrust is negative but the torque remains positive. Thus the propeller is acting as a *brake*.

If $J > J_2$, both thrust and torque are negative and the propeller acts as a *windmill*, i.e. supplies power instead of consuming it. The efficiency is then $1/\eta$ taken positively.

If $J = J_2$ the propeller is capable of *autorotation*, i.e. of rotating without demanding power from the engine, as in the autogyro.

With regard to J_1, if the propeller makes one revolution in a unit of time it advances the distance $V = J_1 D$. The length $J_1 D$ is the *experimental mean pitch*, and is the distance the propeller advances per complete turn of the blades when no thrust is exerted.

In variable pitch propellers it is possible to rotate the blades, each about the blade axis, and thus obtain a different experimental mean pitch for each setting.

This turning of the blades will of course alter the incidence of every blade element.

13·5. Infinitely many blades. If we suppose the propeller to have an infinite number of equal blades each carrying an infinitesimal proportion of the total thrust, the situation undergoes a notable simplification in that the principle of momentum is easily applied. Referring to fig. 13·31 we shall consider the air which occupies the slipstream and its upstream "extension" and bounded by the section AA', GG', the former being so far upstream that the velocity is V parallel to the axis of rotation and the pressure is Π, the latter being a long way downstream at a point where the slipstream has become sensibly cylindrical. To this part of the slipstream we apply the suffix unity so that the quantities p, ω, v are denoted by p_1, ω_1, v_1, and are functions of r only and not of the azimuth θ. We denote by $2R_1$ the diameter of this part of the slipstream. The forces acting on the body of fluid here considered are (i) the thrust T, (ii) the pressure thrust due to uniform pressure Π over AA', (iii) the pressure thrust due to p_1 over GG', (iv) the pressure thrust due to pressure p over the curved boundary AG, $A'G'$. A uniform pressure $-\Pi$ applied over the whole boundary yields no resultant force and, supposing this to be applied, (ii) is eliminated and (iii) and (iv) are due to pressures $p_1-\Pi$ and $p-\Pi$ respectively. If we denote by X the component of the new (iv) in the direction of T, we can equate the resultant force to the net flux of momentum out of the volume $AA'\,GG'$.

Thus we get

$$(1)\quad T + X - \int_0^{R_1}(p_1 - \Pi)\,2\pi r\,dr = \rho\int_0^{R_1} 2\pi r\,dr\,(V+v_1)^2 - \rho V\int_0^{R_1} 2\pi r\,dr\,(V+v_1),$$

the second integral on the right giving the volume flux out of GG' and therefore by the equation of continuity the corresponding flux at AA'. Thus, taking the value of $p_1 - \Pi$ from 13·31 (4) we get

$$(2)\qquad T + X = 2\pi\rho\int_0^{R_1}[r^2\omega_1(\Omega - \tfrac{1}{2}\omega_1) + \tfrac{1}{2}v_1^2]\,dr.$$

In 13·51 we shall prove that $X = 0$.

To evaluate p_1 observe that in the cylindrical part of the slipstream, resolving radially,

$$(3)\qquad \frac{dp_1}{dr} = r\omega_1^2\,\rho,$$

which means that p_1 decreases as we move towards the axis of rotation, and since $p_1 = \Pi$ when $r = R_1$ we have

$$(4)\qquad p_1 - \Pi = -\rho\int_r^{R_1} r\omega_1^2\,dr.$$

13·51. The encased propeller. If we consider the propeller with infinitely many blades to be operating in an infinite coaxial cylindrical tube of diameter $2h$, equation 13·5 (1) will still hold. Outside the cylindrical part of the slipstream the velocity will be constant and equal to, say, $V - v_2'$, the fact that it is less than V following from the equation of continuity. If we consider the air outside the slipstream and its " extension ", we shall have by the same argument as that which yields 13·5 (1)

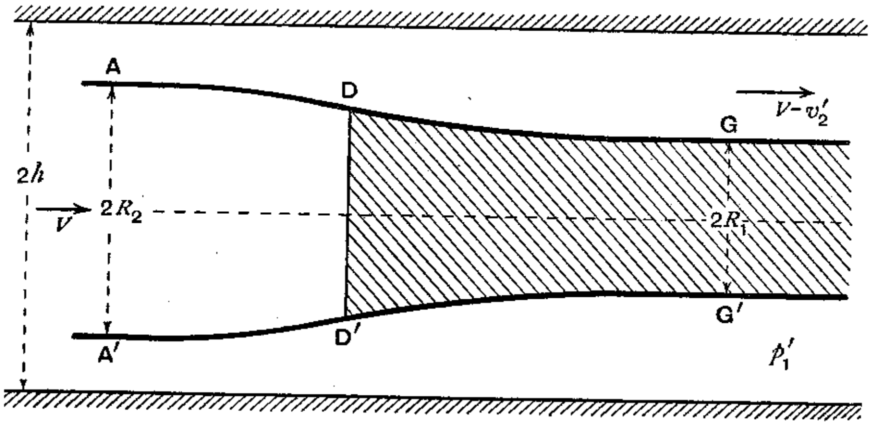

FIG. 13·51.

$$(1) \qquad X + \int_{R_1}^{h} (p_1' - \Pi)\, 2\pi r\, dr = 2\pi\rho \int_{R_1}^{h} (V - v_2') v_2' r\, dr,$$

where p_1' is the pressure over the section outside the slipstream. Now by Bernoulli's theorem (or by 13·31 (4), noting that $\omega = 0$, $v = -v_2'$)

$$p_1 - \Pi = \rho v_2'(V - \tfrac{1}{2} v_2').$$

Therefore from (1), since v_2' is constant,

$$(2) \qquad X = 2\pi\rho \int_{R_1}^{h} -\tfrac{1}{2} v_2'^2\, r\, dr = -\tfrac{1}{2}\pi\rho (h^2 - R_1^2) v_2'^2,$$

and thus it appears that X is negative, i.e. opposes the thrust.

Now by the equation of continuity, if $2R_2$ is the diameter AA' of the extension of the slipstream in fig. 13·51, we have

$$\pi(h^2 - R_2^2)\, V = \pi(h^2 - R_1^2)(V - v_2')$$

and therefore $$v_2' = (R_2^2 - R_1^2)\, V/(h^2 - R_1^2),$$

so that from (2) $$X = -\tfrac{1}{2}\pi\rho V^2 \frac{(R_2^2 - R_1^2)^2}{h^2 - R_1^2},$$

and when $h \to \infty$, i.e. when the casing is absent, $X \to 0$, which proves the assertion made in 13·5.

The problem envisaged in this section may be regarded as approximating to the case of a propeller in a wind tunnel of circular section.

13·6. Froude's law. Consider a propeller with infinitely many blades and introduce

Assumption V. The contraction of the slipstream may be neglected.

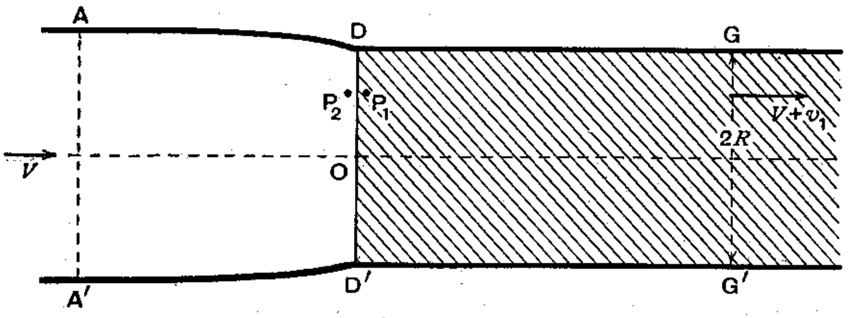

FIG. 13·6.

The diagram now shows a slipstream of radius R, which is cylindrical throughout. As before, AA', GG' represent cross-sections of the "extension" and the slipstream at *infinite* distance from the disc. On GG' at radial distance r the axial velocity will be $V + v_1$ so that v_1 is the velocity induced by the *trailing* vortex system constituting the slipstream. Similarly, ω_1 is the angular velocity induced at the same point by the trailing vortex system. Thus, at a point of GG' each half of the infinite trailing vortex system induces the velocities $\frac{1}{2}v_1$, $\frac{1}{2}\omega_1$. Thus the corresponding velocities induced at a point of the disc at radius r are $\frac{1}{2}v_1$ and $\frac{1}{2}\omega_1$.

This is Froude's law.

13·7. Interference factors. Considering still the propeller with infinitely many blades we introduce

Assumption VI. The induced angular velocity is insufficient to produce appreciable variation of pressure across a section of the slipstream.

With this assumption 13·5 (4) shows that $p_1 = \Pi$.

Now the flux of mass through the annulus of the disc comprised between radii r and $r + dr$ is $2\pi r\, dr\, \rho(V + aV)$. Therefore if dT_r and dQ_r are the thrust and torque on this annulus,

$$dT_r = 2\pi r\, dr \rho(V + aV)v_1, \qquad dQ_r = r\,.\, 2\pi r\, dr \rho(V + aV)\omega_1 r^2,$$

by the principles of linear and angular momentum. Therefore

$$(1) \qquad \frac{dT_r}{dr} = 4\pi r \rho V^2 a(1 + a), \qquad \frac{dQ_r}{dr} = 4\pi r^3 \rho V \Omega (1 + a)a',$$

since from 13·4, $v_1 = 2aV$, $\omega_1 = 2a'\Omega$.

Assumption VII. The formulae (1) can be applied to a propeller with a finite number of blades.

If we equate the values (1) to the corresponding values of 13·41 (3), (4), we get

$$\frac{a}{1+a} = \frac{C_T\,\sigma}{2(1-\cos 2\phi)}, \qquad \frac{a'}{1-a'} = \frac{C_Q\,\sigma}{2\sin 2\phi},$$

and from 13·42 (5)

$$\tan\phi = \frac{1+a}{1-a'}\,\frac{J}{\pi\xi}.$$

These three equations then determine a, a', and ϕ. Graphical methods can be applied to finding the solution.

EXAMPLES XIII

1. If $k_T = T/\rho n^2 D^4$, $k_Q = Q/\rho n^2 D^5$, show that, with the definitions of 13·42,

$$k_T = \frac{\pi^2}{4}\,\tau, \quad k_Q = \frac{\pi^2}{8}\,\kappa.$$

Show also that the efficiency is $Jk_T/2\pi k_Q$.

2. If T, Q, P are the thrust, torque and power of a propeller, show that

$$\frac{1}{\rho}T = \frac{\pi^2\tau}{J^2}R^2V^2 = \frac{\pi^4\tau}{J^4}\,\frac{V^4}{\Omega^2},$$

$$\frac{1}{\rho}Q = \frac{\pi^2\kappa}{J^2}R^3V^2 = \frac{\pi^5\kappa}{J^5}\,\frac{V^5}{\Omega^3},$$

$$\frac{1}{\rho}P = \frac{\pi^3\kappa}{J^3}R^2V^3 = \frac{\pi^5\kappa}{J^5}\,\frac{V^5}{\Omega^2}.$$

3. Prove that the free vortex lines of the absolute motion coincide with the streamlines of the relative motion in the slipstream.

4. Prove that the total circulation round the blade elements at radius r is $\pi\sigma C_L r W$.

5. Show that the loss of energy for the blade elements at radius r is, in unit time,

$$dE = (1-a')\Omega\, dQ_r - (1+a)\,V\,dT_r.$$

Hence prove that

$$dE = \tfrac{1}{2}C_D \rho W^3 Bc\, dr,$$

which is the work done against the drag of the blade elements in unit time.

6. If ϵ is the angle between the apparent and effective relative velocities at the blade elements at radius r, prove that

$$\epsilon = \frac{Bc}{8\pi r}\left(\frac{C_L}{\sin\phi} - \frac{C_D}{\cos\phi}\right).$$

7. If η_r is the efficiency of the blade elements at radius r, defined by

$$\eta_r = \frac{V}{\Omega}\,\frac{dT_r}{dQ_r},$$

prove that

$$\eta_r = \frac{1-a'}{1+a}\,\frac{1-\epsilon\tan\phi}{1+\epsilon\cot\phi}, \text{ where } \epsilon = \frac{C_D}{C_L}.$$

8. If a free vortex of circulation $\Gamma(r)$ issues from a point P, at radial distance r, of the disc and proceeds downstream as a regular helix, prove that, at the point P' of the disc at radial distance r', and at angular distance β from P, the components of velocity induced by the helical vortex are

$$q_a = \frac{\Gamma(r)}{4\pi} r \int_0^\infty l^{-3} [r - r' \cos(\theta + \beta)]\, d\theta,$$

$$q_t = \frac{\Gamma(r)}{4\pi} \frac{V}{\Omega} \int_0^\infty l^{-3} [r' - r\cos(\theta + \beta) - r\theta \sin(\theta + \beta)]\, d\theta,$$

$$q_r = \frac{\Gamma(r)}{4\pi} \frac{V}{\Omega} \int_0^\infty l^{-3} [r \sin(\theta + \beta) - r\theta \cos(\theta + \beta)]\, d\theta,$$

where

$$l^2 = r^2 + r'^2 - 2rr' \cos(\theta + \beta) + V^2\theta^2/\Omega^2.$$

9. Draw graphs to show, for the propeller with infinitely many blades, (i) the axial incremental speed, and (ii) the incremental angular speed ω at radius r, in proceeding from far ahead to far astern of the propeller.

Add to (ii) a graph to show the part due to the bound vortices.

10. In a propeller with infinitely many blades, prove that the pressure on the downstream face of the disc exceeds the pressure on the upstream face by $\rho v_1(V + \frac{1}{2}v_1)$, where v_1 is the axial incremental velocity far down in the slipstream and the other incremental velocities are neglected.

Prove that the efficiency is

$$\left(1 + \frac{v_1}{2V}\right)^{-1}.$$

11. In a propeller with infinitely many blades, prove that the pressure jump in passing through the disc is at radius r

$$\rho r^2 \omega (\Omega - \tfrac{1}{2}\omega),$$

where $r\omega$ is the tangential velocity at radius r.